

A NOTE ON THE (h, q) -ZETA TYPE FUNCTION WITH WEIGHT α

ELIF CETIN, MEHMET ACIKGOZ, ISMAIL NACI CANGUL,
AND SERKAN ARACI

ABSTRACT. The objective of this paper is to derive symmetric property of (h, q) -Zeta function with weight α . By using this property, we give some interesting identities for (h, q) -Genocchi polynomials with weight α . As a result, our applications possess a number of interesting property which we state in this paper.

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1. INTRODUCTION

Recently, T. Kim has developed a new method by using q -Volkenborn integral (or p -adic q -integral on \mathbb{Z}_p) which has added a weight to q -Bernoulli polynomials and investigated their properties (see [8]). He also showed that this polynomials are closely related to weighted q -Bernstein polynomials and derived novel properties of q -Bernoulli numbers with weight α by using symmetric property of weighted q -Bernstein polynomials on the q -Volkenborn integral (for more details, see [10]). After, Araci *et al.* have introduced weighted (h, q) -Genocchi polynomials and so defined (h, q) -Zeta type function with weight by applying Mellin transformation to generating function of (h, q) -Genocchi polynomials with weight α which interpolates for (h, q) -Genocchi polynomials with weight α at negative integers (for details, see [20]). In this paper, we also consider (h, q) -Zeta type function with weight and derive some interesting properties.

We firstly list some notations as follows:

Imagine that p be a fixed odd prime. Throughout this work \mathbb{Z} , \mathbb{Z}_p , \mathbb{Q}_p and \mathbb{C}_p will denote by the ring of integers, the field of p -adic rational numbers and the completion of the algebraic closure of \mathbb{Q}_p , respectively. Also we denote $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ and $\exp(x) = e^x$. Let $v_p : \mathbb{C}_p \rightarrow \mathbb{Q} \cup \{\infty\}$ (\mathbb{Q} is the field of rational numbers) denote the p -adic valuation of \mathbb{C}_p normalized so that $v_p(p) = 1$. The absolute value on \mathbb{C}_p will be denoted as $|\cdot|$, and $|x|_p = p^{-v_p(x)}$ for $x \in \mathbb{C}_p$. When one speaks of q -extensions, q is considered in many ways, e.g. as an indeterminate, a complex number $q \in \mathbb{C}$, or a p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume $|1 - q|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. We use the following

notation

$$(1) \quad [x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}$$

where we want to note that $\lim_{q \rightarrow 1} [x]_q = x$; cf. [1-21].

For a fixed positive integer d , set

$$\begin{aligned} X &= X_d = \varprojlim_n \mathbb{Z}/dp^n\mathbb{Z}, \\ X^* &= \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} a + dp\mathbb{Z}_p \end{aligned}$$

and

$$a + dp^n\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^n}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^n$ (see [1-21]).

The following q -Haar distribution is defined by T. Kim

$$\mu_q(x + p^n\mathbb{Z}_p) = \frac{q^x}{[p^n]_q}$$

for any positive n (see [11], [12]).

Let $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable function on \mathbb{Z}_p . We say that f is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, if the difference quotient

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y}$$

has a limit $f'(a)$ as $(x, y) \rightarrow (a, a)$ and denote this by $f \in UD(\mathbb{Z}_p)$. In [11] and [12], the p -adic q -integral of the function $f \in UD(\mathbb{Z}_p)$ is defined by Kim

$$(2) \quad I_q(f) = \int_{\mathbb{Z}_p} f(\xi) d\mu_q(\xi) = \lim_{n \rightarrow \infty} \sum_{\xi=0}^{p^n-1} f(\xi) \mu_q(\xi + p^n\mathbb{Z}_p)$$

The bosonic integral is considered as the bosonic limit $q \rightarrow 1$, $I_1(f) = \lim_{q \rightarrow 1} I_q(f)$. Similarly, the p -adic fermionic integration on \mathbb{Z}_p is defined by Kim [5] as follows:

$$(3) \quad I_{-q}(f) = \lim_{q \rightarrow -q} I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x)$$

By using fermionic p -adic q -integral on \mathbb{Z}_p , (h, q) -Genocchi polynomials are defined by [20]

$$\begin{aligned} (4) \quad \frac{\tilde{G}_{n+1,q}^{(\alpha,h)}(x)}{n+1} &= \int_{\mathbb{Z}_p} q^{(h-1)\xi} [x + \xi]_{q^\alpha}^n d\mu_{-q}(\xi) \\ &= \lim_{n \rightarrow \infty} \frac{1}{[p^n]_{-q}} \sum_{\xi=0}^{p^n-1} (-1)^\xi [x + \xi]_{q^\alpha}^n q^{h\xi}. \end{aligned}$$

For $x = 0$ in (4), we have $\tilde{G}_{n,q}^{(\alpha,h)}(0) := \tilde{G}_{n,q}^{(\alpha,h)}$ are called (h, q) -Genocchi numbers with weight α which is defined by

$$\tilde{G}_{0,q}^{(\alpha,h)} = 0 \text{ and } q^h \frac{\tilde{G}_{m+1}^{(\alpha,h)}(1)}{m+1} + \frac{\tilde{G}_{m+1}^{(\alpha,h)}}{m+1} = \begin{cases} [2]_q, & \text{if } m = 0, \\ 0, & \text{if } m \neq 0. \end{cases}$$

By (4), we have distribution formula for (h, q) -Genocchi polynomials, which is shown by [20]

$$\tilde{G}_{n+1,q}^{(\alpha,h)}(x) = \frac{[2]_q}{[2]_{q^a}} [a]_{q^a}^n \sum_{j=0}^{a-1} (-1)^j q^{jh} \tilde{G}_{n+1,q^a}^{(\alpha,h)}\left(\frac{x+j}{a}\right).$$

By applying some elementary methods, we shall give symmetric properties of weighted (h, q) -Genocchi polynomials and weighted (h, q) -Zeta type function. Consequently, our applications seem to be interesting and worthwhile for studying in Theory of Analytic Numbers.

2. ON THE (h, q) -ZETA-TYPE FUNCTION

In this part, we firstly recall the (h, q) -Zeta type function with weight α which is derived in [20] as follows:

$$(5) \quad \tilde{\zeta}_q^{(\alpha,h)}(s, x) = [2]_q \sum_{m=0}^{\infty} \frac{(-1)^m q^{mh}}{[m+x]_{q^a}^s}$$

where $q \in \mathbb{C}$, $h \in \mathbb{N}$ and $\Re(s) > 1$. It is clear that the special case $h = 0$ and $q \rightarrow 1$ in (5), it reduces to the ordinary Hurwitz-Euler zeta function. Now, we consider (5) in this form

$$\tilde{\zeta}_{q^a}^{(\alpha,h)}\left(s, bx + \frac{bj}{a}\right) = [2]_{q^a} \sum_{m=0}^{\infty} \frac{(-1)^m q^{mah}}{\left[m + bx + \frac{bj}{a}\right]_{q^{a\alpha}}^s}$$

By applying some basic operations to the above identity, that is, for any positive integers m and b , there exist unique non-negative integers k and i such that $m = bk + i$ with $0 \leq i \leq b-1$. For $a \equiv 1 \pmod{2}$ and $b \equiv 1 \pmod{2}$. Thus, we can compute as follows:

(6)

$$\begin{aligned} \tilde{\zeta}_{q^a}^{(\alpha,h)}\left(s, bx + \frac{bj}{a}\right) &= [a]_{q^a}^s [2]_{q^a} \sum_{m=0}^{\infty} \frac{(-1)^m q^{mah}}{[ma + abx + bj]_{q^{a\alpha}}^s} \\ &= [a]_{q^a}^s [2]_{q^a} \sum_{m=0}^{\infty} \sum_{i=0}^{b-1} \frac{(-1)^{i+mb} q^{(i+mb)ah}}{[(i+mb)a + abx + bj]_{q^{a\alpha}}^s} \\ &= [a]_{q^a}^s [2]_{q^a} \sum_{i=0}^{b-1} (-1)^i q^{iah} \sum_{m=0}^{\infty} \frac{(-1)^m q^{mbah}}{[ab(m+x) + ai + bj]_{q^{a\alpha}}^s} \end{aligned}$$

From this, we can easily discover the following

$$(7) \quad \sum_{j=0}^{a-1} (-1)^j q^{jbh} \tilde{\zeta}_{q^a}^{(\alpha, h)} \left(s, bx + \frac{bj}{a} \right) = [a]_{q^\alpha}^s [2]_{q^a} \sum_{j=0}^{a-1} (-1)^j q^{jbh} \sum_{i=0}^{b-1} (-1)^i q^{iah} \sum_{m=0}^{\infty} \frac{(-1)^m q^{mbah}}{[ab(m+x) + ai + bj]_{q^\alpha}^s}$$

Replacing a by b and j by i in (6) and so we have the following

$$\tilde{\zeta}_{q^b}^{(\alpha, h)} \left(s, ax + \frac{ai}{b} \right) = [b]_{q^\alpha}^s [2]_{q^b} \sum_{j=0}^{a-1} (-1)^j q^{jbh} \sum_{m=0}^{\infty} \frac{(-1)^m q^{mbah}}{[ab(m+x) + ai + bj]_{q^\alpha}^s}$$

By considering the above identity in (7), we can easily state the following theorem.

Theorem 2.1. *The following*

$$\frac{[2]_{q^b}}{[a]_{q^\alpha}^s} \sum_{i=0}^{a-1} (-1)^i q^{ibh} \tilde{\zeta}_{q^a}^{(\alpha, h)} \left(s, bx + \frac{bi}{a} \right) = \frac{[2]_{q^a}}{[b]_{q^\alpha}^s} \sum_{i=0}^{b-1} (-1)^i q^{iah} \tilde{\zeta}_{q^b}^{(\alpha, h)} \left(s, ax + \frac{ai}{b} \right)$$

is true.

Now, setting $b = 1$ in Theorem 2.1, we have the following distribution formula

$$(8) \quad \tilde{\zeta}_q^{(\alpha, h)}(s, ax) = \frac{[2]_q}{[2]_{q^a} [a]_{q^\alpha}^s} \sum_{i=0}^{a-1} (-1)^i q^{iah} \tilde{\zeta}_{q^a}^{(\alpha, h)} \left(s, x + \frac{i}{a} \right).$$

If putting $a = 2$ in (8) leads to the following corollary.

Corollary 2.2. *The following identity holds true:*

$$\tilde{\zeta}_q^{(\alpha, h)}(s, 2x) = \frac{[2]_q}{[2]_{q^2} [2]_{q^\alpha}^s} \left(\tilde{\zeta}_{q^2}^{(\alpha, h)}(s, x) - q^h \tilde{\zeta}_{q^2}^{(\alpha, h)} \left(s, x + \frac{1}{2} \right) \right).$$

Taking $s = -m$ into Theorem 2.1, we have the symmetric property of (h, q) -Genocchi polynomials by the following theorem.

Theorem 2.3. *The following identity*

$$[2]_{q^b} [a]_{q^\alpha}^{m-1} \sum_{j=0}^{a-1} (-1)^j q^{jbh} \tilde{G}_{m, q^a}^{(\alpha, h)} \left(bx + \frac{bj}{a} \right) = [2]_{q^a} [b]_{q^\alpha}^{m-1} \sum_{i=0}^{b-1} (-1)^i q^{iah} \tilde{G}_{m, q^b}^{(\alpha, h)} \left(ax + \frac{ai}{b} \right)$$

is true.

Now also, setting $b = 1$ and replacing x by $\frac{x}{a}$ on the above theorem, we can rewrite the following (h, q) -Genocchi polynomials with weight α .

$$\tilde{G}_{n, q}^{(\alpha, h)}(x) = \frac{[2]_q}{[2]_{q^a}} [a]_{q^\alpha}^{n-1} \sum_{i=0}^{a-1} (-1)^i q^{iah} \tilde{G}_{n, q^a}^{(\alpha, h)} \left(\frac{x+i}{a} \right) \quad (2 \nmid a).$$

Due to Araci *et al.* [20], we develop as follows

$$\begin{aligned} \sum_{n=0}^{\infty} \tilde{G}_{n,q}^{(\alpha,h)}(x+y) \frac{t^n}{n!} &= [2]_q t \sum_{m=0}^{\infty} (-1)^m q^{mh} e^{t[x+y+m]_{q^\alpha}} \\ &= [2]_q t \sum_{m=0}^{\infty} (-1)^m q^{mh} e^{t[y]_{q^\alpha}} e^{(q^{\alpha y} t)[x+m]_{q^\alpha}} \\ &= \left(\sum_{n=0}^{\infty} [y]_{q^\alpha}^n \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} q^{\alpha(n-1)y} \tilde{G}_{n,q}^{(\alpha,h)}(x) \frac{t^n}{n!} \right) \end{aligned}$$

by using Cauchy product, we see that

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^n \binom{n}{j} q^{\alpha(j-1)y} \tilde{G}_{j,q}^{(\alpha,h)}(x) [y]_{q^\alpha}^{n-j} \right) \frac{t^n}{n!}.$$

Thus, by comparing the coefficients of $\frac{t^n}{n!}$, we state the following corollary.

Corollary 2.4. *The following equality holds true:*

$$(9) \quad \tilde{G}_{n,q}^{(\alpha,h)}(x+y) = \sum_{j=0}^n \binom{n}{j} q^{\alpha(j-1)y} \tilde{G}_{j,q}^{(\alpha,h)}(x) [y]_{q^\alpha}^{n-j}.$$

By using Theorem 2.3 and (9), we readily derive the following symmetric relation after some applications.

Theorem 2.5. *The following equality holds true:*

$$\begin{aligned} [2]_{q^b} \sum_{i=0}^m \binom{m}{i} [a]_{q^\alpha}^{i-1} [b]_{q^\alpha}^{m-i} \tilde{G}_{i,q^a}^{(\alpha,h)}(bx) \tilde{S}_{m-i;q^b,h+i-1}^{(\alpha)}(a) \\ = [2]_{q^a} \sum_{i=0}^m \binom{m}{i} [b]_{q^\alpha}^{i-1} [a]_{q^\alpha}^{m-i} \tilde{G}_{i,q^b}^{(\alpha,h)}(ax) \tilde{S}_{m-i;q^a,h+i-1}^{(\alpha)}(b) \end{aligned}$$

where $\tilde{S}_{m;q,i}^{(\alpha)}(a) = \sum_{j=0}^{a-1} (-1)^j q^{ji} [j]_{q^\alpha}^m$.

When $q \rightarrow 1$ into Theorem 2.5, it leads to the following corollary.

Corollary 2.6. *The following identity holds true:*

$$\begin{aligned} \sum_{i=0}^m \binom{m}{i} a^{i-1} b^{m-i} G_i(bx) S_{m-i}(a) \\ = \sum_{i=0}^m \binom{m}{i} b^{i-1} a^{m-i} G_i(ax) S_{m-i}(b) \end{aligned}$$

where $S_m(a) = \sum_{j=0}^{a-1} (-1)^j j^m$ and $G_n(x)$ are called the ordinary Genocchi polynomials which is defined via the following generating function

$$\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2t}{e^t + 1} e^{xt}.$$

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Uludag University, Faculty of Arts and Science, Department of Mathematics, Bursa, Turkey

E-mail address: elifc2@hotmail.com

University of Gaziantep, Faculty of Science and Arts, Department of Mathematics, 27310 Gaziantep, TURKEY

E-mail address: acikgoz@gantep.edu.tr

Uludag University, Faculty of Arts and Science, Department of Mathematics, Bursa, Turkey

E-mail address: ncangul@gmail.com

University of Gaziantep, Faculty of Science and Arts, Department of Mathematics, 27310 Gaziantep, TURKEY

E-mail address: mtsrkn@hotmail.com